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Finite Elements on the Sphere

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The origin of this paper is the need for methods of solving the so-called altimetry–gravimetry problem of physical geodesy (see Svensson [7]) numerically. In the spherical approximation this problem leads to a pseudodifferential equation on a sphere, involving an invariant (with respect to the Riemannian geometry of the sphere) pseudodifferential operator of order one on the sphere. In trying to apply the Galerkin method with such an operator, it is natural to use trial functions χ_P which are of axial symmetry around an axis through the point P. The matrix elements to be constructed in the Galerkin method are then of the form $(A\chi_P, \chi_Q)$ and depend, because of the invariance of A and the symmetry of the trial functions, only upon the distance from P to Q. To be able to treat the mixed problem, "small" support for the trial functions is needed. Trial functions χ_P satisfying the requirements above are suggested. Approximation theorems are proved. To get approximations compareable to those of the plane finite element approach one has to let the grid size be a power h^β, $\beta > 1$, of the element diameter h. Efforts of estimating β are made. An analysis of a plane approximation indicates, however, that those estimates are suboptimal. Computation of the matrix elements is discussed.

1. INTRODUCTION

Any boundary value problem

$$\begin{aligned} \Delta u = 0 \qquad & \text{outside (inside) } S, \\ a \cdot \partial u/\partial n + b \cdot u = f \qquad & \text{on } S, \end{aligned} \tag{1.1}$$

where Δ is the Laplace operator, S the unit sphere in three-dimensional space, and a and b real constants with $ab \neq 0$, may be written as a pseudodifferential equation

$$Au = f \tag{1.2}$$

on S, where

$$A = -a(-\Delta_S + 1/4)^{1/2} + (b - a/2). \tag{1.3}$$

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In (1.3) Δ_S denotes the Laplacian on S. We refer, e.g., to Hörmander [4] and Svensson [7]. Examples are the Dirichlet problem, the Neumann problem, and the Stokes' problem of physical geodesy—the determination of the disturbed potential from gravity anomalies. In the last example the primary interest lies in determining u on S itself and, in any case, when u is determined on S it may be continued by the classical Dirichlet formula.

Mixed problems

$$\begin{gathered} \Delta u = 0 \qquad \text{outside(inside)} S, \\ a \cdot \partial u / \partial n + b \cdot u = f \qquad \text{on } \Omega, \\ u = v \qquad \text{on } \Omega', \end{gathered} \tag{1.4}$$

where S is split by an infinitely smooth curve Γ into two open sets Ω and Ω', may be reduced to an equation (see Èskin [1]),

$$p_\Omega A u = f, \tag{1.5}$$

where p_Ω denotes the restriction operator to Ω. One example is the efforts of combining satellite measurements over the oceans (Ω') with terrestrial data from the continents (Ω) in the so called altimetry–gravimetry technique of physical geodesy (see Svensson [7]).

Considering (1.2) for $f \in \mathscr{H}_{s-1}(S)$ and $u \in \mathscr{H}_s(S)$ and (1.5) for $f \in \bar{\mathscr{H}}_{t-1}(\Omega)$ and $u \in \mathring{\mathscr{H}}_t(\Omega)$, where $0 < t < 1$, it follows from the theory of elliptic pseudodifferential operators that, in both cases the Fredholm theory applies. Here $\mathscr{H}_s(S)$ denotes the Sobolev space of order s on S, $\bar{\mathscr{H}}_s(\Omega)$ consists of all distributions on Ω which admit extensions $l(g)$ to elements in $\mathscr{H}_s(S)$ with norm $\inf \|l(g)\|_s$, the infimum being taken over all extensions, and $\mathring{\mathscr{H}}_s(\Omega)$ is the set of all elements in $\mathscr{H}_s(S)$ with support in Ω and the induced norm.

In more refined models, such as in the Molodensky problem of physical geodesy (see Heiskanen and Moritz [2]) the operator A of (1.3) is replaced by a more complicated operator which still is in some sense close to the original A.

The pseudodifferential operator of (1.3) is of order 1, it is elliptic, and it is invariant (for mappings of S onto itself which preserve the Riemann metric). When trying to solve (1.2) or (1.4) numerically by Galerkin's method it is natural to try to use trial functions of axial symmetry, since then the matrix elements $(A\chi, P)$ will be comparatively easy to compute. In case we use one single trial function of axial symmetry but vary the axis, or, equivalently, the pole P, the matrix element $(A\chi_P, \chi_Q)$ will depend only upon the *distance* between the poles P and Q in case A is invariant. Furthermore it is in at least (1.5) necessary to use trial functions of small support. Hence we are led to try to construct some kind of finite elements on the sphere by aid of base functions of axial symmetry and support in a cap. Of course one should not expect to get the remarkable convergence rates of the plane finite element approach using roof functions. Indeed, if we use a grid of points on S, being in some sense uniformly distributed and with a characteristic mesh size h, then we cannot in general take the supports of the base functions to have a diameter proportional to h but rather to $h^{1/\beta}$, where β is some number larger than 1. The approximation error does then satisfy

$$\|u - u_h\|_s \leqslant Ch^{\gamma/\beta} \|u\|_{s+\gamma}.$$

The value of β will be discussed in Section 2. The base functions χ_P are defined by

$$\chi_P(\psi) = \frac{\cos\psi - \cos\psi_h}{1 - \cos\psi_h}, \qquad \text{for} \quad 0 \leqslant \psi \leqslant \psi_h,$$
$$= 0, \qquad \text{otherwise},$$

where ψ denotes angular distance to the point P and where ψ_h is a small positive number.

The standard technique of the Galerkin approximation applies, employing positivity or ellipticity of A. We have only to consider how good a function in $\mathscr{H}_s(S)$ may be approximated by a linear combination of base functions. Such approximations are discussed in Section 2 while the computation of matrix elements is discussed in Section 3. Despite the addition of difficulties in the approximation, the computation of the matrix elements is so simple that the method should be competitive at least in mixed problems.

Concerning notation, C will denote a positive constant, the value of which, however, will vary from formula to formula. The norm in the Sobolev spaces $\mathscr{H}_s(S)$ and $\mathscr{H}_s(R^2)$, respectively, will be denoted by $\| \; \|_s$ and be defined by

$$\|u\|_s = \|(-\Delta_S + \tfrac{1}{4})^{s/2} u\|_0$$

and

$$\|u\|_s = \|(-\Delta + 1)^{s/2} u\|_0,$$

where $\| \; \|_0$ denotes the L_2-norm and Δ_S and Δ the respective Laplacians. Further we shall write

$$(u, v) = \iint_S u\bar{v}\, dS.$$

2. The Approximation

Consider the trial functions (1.6) with a given ψ_h. We shall choose a grid of points P such that

$$u_h = c^{-1} \sum_P (u, \chi_P) \chi_P, \tag{2.1}$$

where

$$c = \pi N \psi_h^4/4, \qquad (u, \chi_P) = \iint_S u\chi_P \, dS,$$

and N is the number of grid points, approximates u. It will be convenient to use the normalized surface harmonics

$$Y_{nm}(\theta, \lambda) = c_{nm} P_n^m(\cos\theta)\, e^{im\lambda}, \qquad n \geqslant 0, \, |m| \leqslant n,$$

where P_n^m are the Legendre functions of degree n and order m, and where c_{nm} is choosen to give $\|Y_{nm}\|_0 = 1$. Further we shall write $Y_n(\theta) = Y_{n0}(\theta, \lambda)$ and also $Y_{nm}(Q) = Y_{nm}(\theta, \lambda)$ if θ, λ are the spherical coordinates of Q in a standard reference system. By ψ_{PQ} we shall mean the angular distance from P to Q. The addition theorem for surface harmonics reads

$$Y_n(\psi_{PQ}) = (4\pi/(2n+1))^{1/2} \sum_{|m| \leqslant n} Y_{nm}(Q)\, \overline{Y_{nm}(P)}. \tag{2.2}$$

The expansions of u and χ_P reads

$$u(Q) = \sum_{n \geqslant 0} \sum_{|m| \leqslant n} u_{nm} Y_{nm}(Q), \tag{2.3}$$

$$\chi_P(Q) = \sum_{n \geqslant 0} a_n Y_n(\psi_{PQ}). \tag{2.4}$$

Formula (2.1) may be rewritten as

$$u_h(Q) = \iint_S u(Q')\, K_h(Q, Q')\, dS(Q'), \tag{2.5}$$

where

$$K_h(Q, Q') = c^{-1} \sum_P \chi_P(Q)\, \chi_P(Q'). \tag{2.6}$$

Introduce also the kernel

$$L_h(Q, Q') = N/(4\pi c) \iint_S \chi_P(Q)\, \chi_P(Q')\, dS(P) \tag{2.7}$$

and

$$\hat{u}_h = \iint_S L_h u \, dS. \tag{2.8}$$

A straightforward computation employing the addition theorem (2.2) yields

$$L_h(Q, Q') = N/(4\pi c) \cdot \sum_{n \geqslant 0} (4\pi/(2n+1))^{1/2} a_n^2 Y_n(\psi_{QQ'}),$$

and hence

$$\hat{u}_h = \sum_{n \geqslant 0} \sum_{|m| \leqslant n} (N/4\pi c)(4\pi/(2n+1)) a_n^2 u_{nm} Y_{nm}.$$

Consequently, if we choose the norm $\|v\|_s = \|(-\Delta_S + 1/4)^{s/2} v\|_0$ on $\mathscr{H}_s(S)$, where $\| \, \|_0$ denotes the L_2-norm, we get

$$\|u - \hat{u}_h\|_s^2 = \sum_{n \geqslant 0} \sum_{|m| \leqslant n} (n + 1/2)^{2s} (1 - Na_n^2/c(2n+1))^2 |u_{nm}|^2. \tag{2.9}$$

Letting P_n denote the Legendre polynomial of degree n when $n \geqslant 0$ and putting $P_n = 1$ for $n < 0$, a straightforward computation, employing the well-known formula $P'_{n+1} - P'_{n-1} = (2n+1) P_n$, yields for $n \geqslant 0$,

$$a_n = (\pi/(2n+1))^{1/2} (1-t)^{-1} \left(\frac{P_{n+2}(t) - P_n(t)}{2n+3} - \frac{P_n(t) - P_{n-2}(t)}{2n-1} \right), \tag{2.10}$$

where $t = \cos(\psi_h)$. From (2.10) we find that, as a function of t, $b_n = (1-t) a_n$ satisfies, $b_n(1) = b'_n(1) = 0$, $b''_n = (\pi)^{1/2} (2n+1)^{1/2} P_n(t)$. Hence

$$|a_n| \leqslant (\pi(2n+1))^{1/2} (1-t)/2 \qquad \text{for all } n \text{ and } t, \tag{2.11}$$

and since $|1 - P_n(\cos \psi)| \leqslant C(n\psi)^2$ (see Appendix)

$$|a_n - (\pi(2n+1))^{1/2} (1-t)/2| \leqslant Cn^{5/2}(1-t)^2. \tag{2.12}$$

We separate in the sum of (2.9) the terms where $n\psi_h > 1$ from those with $n\psi_h \leqslant 1$. In the first case we have in view of (2.11)

$$\sum_{n > 1/\psi_h} \sum_{|m| \leqslant n} (n + 1/2)^{2s} (1 - Na_n^2/(c(2n+1))^2) |u_{nm}|^2$$
$$\leqslant C \sum_{n > 1/\psi_h} \sum_{|m| \leqslant n} (n + 1/2)^{2s} |u_{nm}|^2 \leqslant C\psi_h^{2\gamma} \|u\|_{s+\gamma}^2,$$

while in the second, by (2.11) and (2.12),

$$|1 + 4a_n^2/(\pi(2n+1)(1-t)^2| \leqslant Cn^2(1-t) \leqslant Cn^2\psi_h^2. \tag{2.13}$$

Hence, for $\gamma \leqslant 2$ we get

$$\|u - \hat{u}_h\|_s^2 \leqslant C\psi_h^{2\gamma} \|u\|_{s+\gamma}^2, \tag{2.14}$$

where C is independent of u and ψ_h. Now we shall estimate $\|u_h - \hat{u}_h\|_s$. In order that the discrete kernel K_h shall approximate L_h we make the following requirement upon the grid; assuming β and ψ_h to be given:

CONDITION A. There is a partition of S into N (= the number of points in the grid) mutually disjoint parts A_P of equal area with the property that each grid point may be ordered to a unique part A_P of the sphere in such a way that the maximal distance d_P from P to any point of A_P satisfies

$$d_P \leqslant C\psi_h^\beta,$$

where C is independent of ψ_h.

Assuming Condition A we may write

$$\hat{u}_h - u_h = (u, V), \tag{2.15}$$

where

$$V(Q, Q') = (N/4\pi c) \cdot \sum_P \iint_{A_P} (\chi_{P'}(Q)\,\chi_{P'}(Q') - \chi_P(Q)\,\chi_P(Q'))\, dS(P').$$

By applying the triangle and Cauchy inequalities to (2.15) we get

$$\|\hat{u}_h - u_h\|_s \leqslant C\psi_h^{-4} \sup(K)\, \|u\|_{s+\gamma}, \tag{2.16}$$

where

$$K = \|\chi_{P'} - \chi_P\|_s \, \|\chi_P\|_{-s-\gamma} + \|\chi_{P'} - \chi_P\|_{-s-\gamma} \|\chi_P\|_s,$$

and the supremum is to be taken over all P', P with a mutual distance $d(P, P') < C\psi_h^\beta$ (cf. Condition A). Now

$$\|\chi_P\|_s^2 = \sum_{n \geqslant 0} (n + 1/2)^{2s}\, a_n^2$$

and, by (2.4) and the addition theorem (2.2), choosing P as pole,

$$\begin{aligned}(\chi_{P'} - \chi_P)(Q) &= \sum_{n \geqslant 0} a_n (Y_n(\psi_{P'Q}) - Y_n(\psi_{PQ})) \\ &= \sum_{n \geqslant 0} a_n \sum_{|m| \leqslant n} g_{nm} \overline{Y_{nm}(Q)},\end{aligned}$$

where

$$g_{nm} = \delta_{0m} - (4\pi/(2n+1))^{1/2}\, Y_{nm}(P').$$

Hence

$$\|\chi_{P'} - \chi_P\|_t^2 = \sum_{n \geqslant 0} (n+1/2)^{2t}\, a_n^2 \cdot \sum_{|m| \leqslant n} |g_{nm}|^2.$$

But

$$|1 - (4\pi/(2n+1))^{1/2}\, Y_{n0}(P')| = 1 - P_n(\cos \psi_{PP'})$$

and, by the addition theorem,

$$\begin{aligned}(4\pi/(2n+1)) \cdot \sum_{|m| \leqslant n} |Y_{nm}(P')|^2 &= (4\pi/(2n+1))^{1/2}\, Y_n(1)\\ &= 1.\end{aligned}$$

Hence

$$\|\chi_{P'} - \chi_P\|_t^2 = \sum_{n \geqslant 0} (n+1/2)^{2t}\, a_n^2\, 2(1 - P_n(\cos \psi_{PP'})). \tag{2.17}$$

For $0 \leqslant \alpha \leqslant 1$ we have (see Appendix) $0 \leqslant 1 - P_n(\cos \psi) \leqslant C(n\psi)^{2\alpha}$ and hence

$$\|\chi_{P'} - \chi_P\|_t \leqslant C\psi_h^{\alpha\beta}\, \|\chi_P\|_{t+\alpha} \qquad \text{if } t + \alpha < 3/2 \text{ and } d(P', P) < C'\psi_h^{\beta}. \tag{2.18}$$

For the norms of χ_P we have (see Appendix) for any s with $-1 < s < \frac{3}{2}$

$$\|\chi_P\|_s \leqslant C\psi_h^{1-s}. \tag{2.19}$$

Estimating the terms of K of (2.16) by aid of (2.18)–(2.19) both terms are of order $\psi_h^{\alpha\beta+2-\alpha+\gamma}$, assuming that $-1 < s < \frac{3}{2}$ and $s + \gamma < 1$. Hence $\psi_h^{-4-\gamma}K$ is bounded if $\alpha\beta - \alpha \geqslant 2$ or, since $\alpha > 0$,

$$\beta \geqslant 1 + 2/\alpha.$$

Consequently, if s in (2.16) satisfies $s < \frac{1}{2}$ we may choose $\alpha = 1$, in which case any $\beta \geqslant 3$ will do. If $s \geqslant \frac{1}{2}$ we may choose as α any number $\alpha < \frac{3}{2} - s$ and, consequently, as β any number $\beta > 1 + 4/(3 - 2s)$. Hence we have proved

THEOREM 2.1. *With trial functions χ_P as defined in Section* 1 *and for* $0 \leqslant s < 1$ *there are values of β such that for any ψ_h and any grid satisfying Condition* A.

$$\|u - u_h\|_s \leqslant C\psi_h^{\gamma}\, \|\gamma\|_{s+\gamma}, \qquad s + \gamma < 1,$$

where C is independent of u, ψ_h, and the specific grid. If $s < \frac{1}{2}$ one such value is $\beta = 3$. If $s \geqslant \frac{1}{2}$, any $\beta > 1 + 4/(3-2s)$ will do.

The value of β may be somewhat improved by replacing χ_P by χ_P^2 in (2.1) with $c = \pi N(1-t)^2/9$. The coefficients in the expansion of χ_P^2 are given, emplying the mean value theorem, by

$$b_n = (\pi(2n+1))^{1/2}\, P_n(\eta t)(1-t)/3,$$

where $0 \leqslant \eta \leqslant 1$. Hence we have immediate counterparts of (2.11) and (2.12), so the counterpart of (2.14) follows. We also note that, since $0 \leqslant 1 - P_n(\cos\psi) \leqslant C(n\psi)^2$, we get from (2.17) a counterpart of (2.18)

$$\|\chi_{P'}^2 - \chi_P^2\|_t \leqslant C\psi_h^\beta \|\chi_P^2\|_{t+1} \qquad \text{for} \quad t < \tfrac{3}{2} \text{ and } d(P', P) \leqslant C'\psi_h^\beta.$$

Now (2.19) is improved to (see Appendix)

$$\|\chi_P^2\|_s \leqslant C\psi_h^{1-s} \qquad \text{for } -1 \leqslant s < \tfrac{5}{2}.$$

Hence, in estimating the supremum of (2.16) for this case we see that both terms are of order $\psi_h^{\beta+1+\gamma}$, and hence $\beta = 3$ will yield the desired estimate.

THEOREM 2.2. *Assume that we replace, in (2.1) χ_P by χ_P^2 and c by $\pi N(1-t)^2/9$. Let $0 \leqslant s \leqslant 1$ and $\beta \geqslant 3$. Then, for any ψ_h and any grid satisfying Condition* A

$$\|u - u_h\|_s \leqslant C\psi_h^\gamma \|u\|_{s+\gamma} \qquad \text{for} \quad s + \gamma < 1,$$

where C is independent of u, ψ_h, and the specific grid.

Obviously the result cannot be improved by the above method of proof by choosing higher powers of χ_P. The figures for β as given in Theorems 2.1–2.2 are, however, most certainly too pessimistic. We consider a plane approximation with trial functions

$$\begin{aligned} \rho_P &= 1 - (r_p/h)^2, \qquad && \text{for } 0 \leqslant r_P \leqslant h, \\ &= 0, && \text{otherwise,} \end{aligned}$$

where r_P denotes the Euclidean distance to the point P. Following Èskin [1] we shall study the approximation by sums of type (2.1) but we shall consider a regular grid in the plane of mesh size h^β. Denoting by $\tilde{f}$ the Fourier transform of f we have

$$\tilde{\rho}_P = h^2 e^{i(p\cdot\xi)h}\tilde{\phi}(h\xi), \tag{2.20}$$

where ph^β is the coordinate vector of P and ϕ equals ρ_P for the case P is the origin and $h=1$. The Parseval equality yields, when employed on u_h as defined by

$$u_h = c^{-1} \sum_P (u, \rho_P)\, \rho_P,$$

where $c = (\pi/2)^2\, h^{4-2\beta}$ and $(u, v) = \iint uv\, dx$,

$$\tilde{u}_h(\xi) = \sum_P (2\pi)^{-2}\, c^{-1} \iint \tilde{u}(\zeta)\, \tilde{\rho}_P(\zeta)\, d\zeta \cdot \tilde{\rho}_P(\xi),$$

or, inserting (2.20),

$$\tilde{u}_h(\xi) = \sum_P h^4 (2\pi)^{-2}\, c^{-1} \iint \tilde{u}(\zeta)\, e^{i(p\cdot(\xi-\zeta))h^\beta} \tilde{\phi}(h\zeta)\, d\zeta \cdot \tilde{\phi}(h\xi).$$

The Poisson summation formula yields, inserting c,

$$\tilde{u}_h(\xi) = \sum_P (2/\pi)^2\, \tilde{u}(\xi - 2\pi p h^{-\beta})\, \tilde{\phi}(h\xi - 2\pi p h^{1-\beta})\, \tilde{\phi}(h\xi). \qquad (2.21)$$

We write

$$\tilde{u}_h(\xi) = \tilde{U}_0(\xi) + \tilde{U}_1(\xi),$$

where $\tilde{U}_0$ is the term with P equal to the origin in (2.21). Since (see Magnus, Oberhettinger, and Soni [5])

$$\tilde{\phi}(\xi) = 2\pi\, |\xi|^{-2}\, (2J_1(|\xi|)\, |\xi|^{-1} - J_0(|\xi|))$$

with the Bessel functions J_0, J_1, it follows that

$$|\tilde{\phi}(\xi) - \pi/2| \leqslant C\, |\xi|, \; |\tilde{\phi}(\xi)| \leqslant C(1 + |\xi|^2)^{-5/4}.$$

Hence, for $\gamma \leqslant 2$

$$\begin{aligned}(1 + |\xi|)^{2s}\, |\tilde{u}(\xi) - \tilde{U}_0(\xi)|^2 &= (1 + |\xi|)^{2s}\, |\tilde{u}(\xi)|^2\, (1 - (2/\pi)^2\, \tilde{\phi}(h\xi)^2)^2 \\ &\leqslant C \cdot h^{2\gamma} (1 + |\xi|)^{2(s+\gamma)}\, |\tilde{u}(\xi)|^2. \qquad (2.22)\end{aligned}$$

Up till now we have almost literally followed Èskin [1] who considers square roof functions instead of our ρ_P: s (which is of course the only reasonable approach in the plane case), but now we have to make deviations. First we observe that

$$\sum_{P \neq 0} \tilde{\phi}(h\phi - 2\pi p h^{1-\beta})^2$$

is bounded. Hence, by the Cauchy inequality

$$\left| \sum_{P \neq 0} \tilde{u}(\xi - 2\pi p h^{-\beta}) \, \tilde{\phi}(h\xi - 2\pi p h^{1-\beta}) \right|^2 \leqslant C \cdot \sum_{P \neq 0} |\tilde{u}(\xi - 2\pi p h^{1-\beta})|^2,$$

and, consequently, by (2.21)

$$\| \tilde{U}_1 \|_s^2 \leqslant C \sum_{p \neq 0} \iint (1 + |\xi|)^{2s} \, |\tilde{u}(\xi - 2\pi h^{1-\beta})|^2 \, \tilde{\phi}(h\xi)^2 \, d\xi.$$

After changing coordintates in each term we get

$$\| \tilde{U}_1 \|_s^2 \leqslant C \iint |\tilde{u}(\zeta)|^2 \, (1 + |\zeta|)^{2(s+\gamma)} \, Q(\zeta) \, d\zeta \cdot h^{2\gamma},$$

so if Q is bounded, we get $\| \tilde{U}_1 \|_s \leqslant Ch^\gamma \| u \|_{s+\gamma}$, which together with (2.22) gives $\| u - u_h \|_s \leqslant Ch^\gamma \| u \|_{s+\gamma}$. We have

$$Q = \sum_{P \neq 0} \tilde{\phi}(h\zeta + 2\pi p h^{1-\beta})^2 \, (1 + |\zeta + 2\pi p h^{-\beta}|)^{2s} \, (1 + |\zeta|)^{-(2s+2\gamma)} \, h^{-2\gamma}.$$

Assume that $0 \leqslant s < \frac{3}{2}$. Clearly Q is uniformly bounded for $|h\zeta| \geqslant 1$. When $|h\zeta| \leqslant 1$, the term in Q is of the magnitude, recalling that $|\tilde{\phi}(\xi)| \leqslant C(1 + |\xi|^2)^{-5/4}$.

$$|p|^{(-5+2s)} \, h^{(1-\beta)(-5+2s)-2(\gamma+s)},$$

and hence Q is bounded provided that $0 \leqslant s < \frac{3}{2}$ and

$$\beta \geqslant (5 + 2\gamma)/(5 - 2s).$$

We collect the result in a theorem.

THEOREM 2.3. *Consider the plane approximation as given above. Let* $0 \leqslant s < \frac{3}{2}$, $0 \leqslant \gamma \leqslant 2$, *and* $\beta \geqslant (5 + 2\gamma)/(5 - 2s)$. *Then*

$$\| u - u_h \|_s \leqslant Ch^\gamma \| u \|_{s+\gamma},$$

where C is independent of u and h.

3. Computing the Matrix Elements

When computing a Galerkin approximation of a solution of $Au = f$, where A is a first order pseudodifferential operator on S, one has to compute the matrix elements

$$(A\chi_P, \chi_Q).$$

Clearly the matrix elements are functions of only the distance between P and Q in case A is invariant. If

$$A = c(-\Delta_S + 1/4)^{1/2} + d$$

and if, ψ_P denoting angular distance to P,

$$\chi_P = \sum_{n \geqslant 0} a_n Y_n(\psi_P)$$

we get

$$(A\chi_P, \chi_Q) = \sum_{n \geqslant 0} \sum_{\nu \geqslant 0} (c(n+1/2)+d)\, a_n a_\nu (Y_n(\psi_P), Y_n(\psi_Q)).$$

By the addition theorem (2.2) we get

$$Y_\nu(\psi_Q) = (4\pi/(2\nu+1))^{1/2} \sum_{|\mu| \leqslant \nu} \overline{Y_{\nu\mu}(\psi_P)}\, Y_{\nu\mu}(\psi_{PQ}).$$

Using the orthogonality we get

$$(Y_n(\psi_P), Y_\nu(\psi_Q)) = \delta_{n\nu}(4\pi/(2n+1))^{1/2}\, Y_n(\psi_{PQ}),$$

i.e.,

$$(A\chi_P, \chi_Q) = \sum_{n \geqslant 0} (2\pi)^{1/2}\, (c(n+1/2)^{1/2} + d(n+1/2)^{-1/2})\, a_n^2 Y_n(\psi_{PQ}). \tag{3.1}$$

Since $Y_n(\psi_{PP}) = ((2n+1)/4\pi)^{1/2}$ the diagonal elements satisfy

$$(A\chi_P, \chi_P) = c \cdot \|\chi_P\|_{1/2}^2 + d \cdot \|\chi_P\|_0^2,$$

and are hence of order ψ_h for small ψ_h, a fact which should be remembered when scaling the equations. For the norms of χ_P we have (see Appendix) with $t = \cos \psi_h$

$$\|\chi_P\|_1^2 = 2\pi, \qquad \|\chi_P\|_0^2 = 2\pi(1-t)/3.$$

Hence, for $0 \leqslant s \leqslant 1$, the Hölder inequality yields

$$\|\chi_P\|_s^2 \leqslant \|\chi_P\|_0^{1-s}\, \|\chi_P\|_1^s \leqslant 2\pi((1-t)/3)^{1-s}. \tag{3.2}$$

Differentiating (3.1) with respect to ψ_{PQ} and using the fact (see Hörmander [3, Lemma 1.3.1]) that

$$|\partial_{\psi_{PQ}} Y_n(\psi_{PQ})| \leqslant Cn^{3/2}$$

we get

$$|\partial_{\psi_{PQ}}(A\chi_P, \chi_Q)| \leqslant C(\|\chi_P\|_1^2 + \|\chi_P\|_{1/2}^2).$$

In view of (3.2) it follows that

$$|\partial_{\Psi_{PQ}}(A\chi_P, \chi_Q)| \leqslant C. \tag{3.3}$$

The conclusion is that if we use (3.1) for computation of $(A\chi_P, \chi_Q)$ as a function of ψ_{PQ} for a discrete set of values, then the error of performing a simple linear interpolation for intermediate values is bounded by a constant times the maximal interval length, independently of ψ_h.

By aid of (3.1) $(A\chi_P, \chi_Q)$ may be computed for a discrete set of values of ψ_{PQ} and intermediate values may be found by interpolation. The coefficients a_n of (3.1) are easily computed. Indeed, using the well-known formula

$$(2n+1)\,P_n = P'_{n+1} - P'_{n-1}$$

we get ($P_n = 1$ for $n < 0$)

$$a_n = (\pi)^{1/2}\,(2n+1)^{1/2}\,(1-t)^{-2}\left(\frac{P_{n+2}(t) - P_n(t)}{2n+3} - \frac{P_n(t) - P_{n-2}(t)}{2n-1}\right).$$

We shall, finally, estimate the error of truncating the series of (3.1). The error of truncation at $n = M$ is bounded by

$$\begin{aligned}\left|\sum_{n>M} (c(n+1/2)+d)\,a_n^2\right| &\leqslant \sum_{n>M} (|c|\,(n+1/2)^2\,a_n^2(n+1/2)^{-1} \\ &\qquad + |d|\,(n+1/2)^2\,a_n^2(n+1/2)^{-2}) \\ &\leqslant (M+1/2)^{-1}(|c|\,\|\chi_P\|_1^2 + (M+1/2)^{-1}\,|d|\,\|\chi_P\|_1^2).\end{aligned}$$

Hence, in view of (3.2) the truncation error ε_M is

$$\varepsilon_M \leqslant (M+1/2)^{-1}\,(|c| + (M+1/2)^{-1}\,|d|)\,2\pi.$$

Appendix

1. *Estimating* $1 - P_n$

We only have to check the estimate

$$|1 - P_n(\cos\psi)| \leqslant C(n\psi)^2 \tag{A1}$$

for small $n\psi$. The result follows from an asymptotic formula by Hilb, generalized by Szegö [8]. We have

$$P_n(\cos\psi) = \sum_{\nu=0}^{\infty} a_\nu(\psi)\, \psi^\nu J_\nu(z_n)/z_n^\nu, \tag{A2}$$

where $z_n = (n+1/2)\psi$, a_ν are regular, and the series uniformly convergent for $0 \leqslant \psi \leqslant 2\ (2^{1/2}-1)\pi - \varepsilon$ for any $\varepsilon > 0$. Here $1 - a_0 = 1 - (\psi/\sin\psi)^{1/2}$ vanishes of order 2 and $a_1 = (\psi/\sin\psi)^{1/2}\,(\psi\cos\psi - \sin\psi)/8\psi\sin\psi$ vanishes of order 1 at $\psi = 0$. Further $J_0 - 1$ vanishes of order 2 and J_ν of order ν at $z_n = 0$. Hence (A1) follows from (A2).

2. *Computing the Norms of* χ_P

By a straightforward computation we get ($t = \cos\psi_h$)

$$\|\chi_P\|_0 = (2\pi/3)^{1/2}\,(1-t)^{1/2}, \qquad \|\chi_P^2\|_0 = (2\pi/5)^{1/2}\,(1-t)^{1/2}.$$

If f is a smooth function, depending only upon polar distance θ we get

$$\begin{aligned}\|f\|_1^2 &= ((-\Delta_S + 1/4)^{1/2} f, (-\Delta_S + 1/4)^{1/2} f)\\ &= (((-\Delta_S + 1/4) f, f)\\ &= ((-\partial^2/\partial\theta^2 - \cot\Theta\,\partial/\partial\Theta) f, f) + \tfrac{1}{4}(f, f).\end{aligned}$$

Hence integration by parts yields

$$\|f\|_1^2 = \|\partial f/\partial\Theta\|_0^2 + \tfrac{1}{4}\,\|f\|_0^2.$$

By a density argument this applies even if f is only in $\mathscr{H}_1(S)$. Hence we compute easily

$$\|\chi_P\|_1 = (\pi/2)^{1/2}\,(3+t)^{1/2}.$$

It follows also immediately that $\|\chi_P^2\|_1 \leqslant C$ and $\|\chi_P^2\|_2 \leqslant C(1-t)^{-1/2}$.

We want to prove

$$\|\chi_P\|_s \leqslant C\psi_h^{1-s} \qquad \text{for} \quad -1 < s < \tfrac{3}{2}. \tag{A3}$$

When studying $\|\chi_P\|_s$ we shall use the invariance of the Sobolev spaces under smooth and regular changes of coordinates, see, e.g., Èskin [1]. Near P we introduce rectangular coordinates $x_1 = \psi\cos(\alpha)$, $x_2 = \psi\sin(\alpha)$, where ψ is angular distance to P and α azimuth at P. Hence, instead of the $\mathscr{H}_s$-norms on S used earlier we may use the $\mathscr{H}_s$-norms in the plane. Now we also observe that the behaviour of the norms $\|\chi_P\|_s$ for varying ψ_h depends entirely upon the jump of $\partial x_P/\partial\psi$ at $\psi=\psi_h$. Hence we may replace χ_P by the function

$$\begin{aligned} \rho_P &= 1-(r/\psi_h)^2, && \text{for} \quad r=(x_1^2+x_2^2)^{1/2}<\psi_h, \\ &= 0, && \text{otherwise,} \end{aligned}$$

which, as χ_P, has a jump of order ψ_h^{-1} for $\partial\rho_P/\partial\psi$ at $\psi=\psi_h$. But (cf. (2.20) with $P=0$)

$$\tilde{\rho}_P(\xi)=\psi_h^2\tilde{\phi}(\psi_h\xi),$$

where $\tilde{\phi}$ is a regular function and $|\phi(\xi)|\leqslant C(1+|\xi|^2)^{-5/4}$. Hence using the $\mathscr{H}_s$-norm in the plane

$$\begin{aligned} \|\rho_P\|_s^2 &= \psi_h^4\iint(1+|\xi|^2)^s\,\tilde{\phi}(\psi_h\xi)^2\,d\xi \\ &= \psi_h^{2(1-s)}\iint(\psi_h^2+|\zeta|^2)^s\,\tilde{\phi}(\zeta)^2\,d\zeta. \end{aligned}$$

For $s>-1$ we have

$$\left|\iint_{|\zeta|\leqslant 1}(\psi_h^2+|\zeta|^2)^s\,\tilde{\phi}(\zeta)^2\,d\zeta\right|\leqslant C\int_0^1(\psi_h^2+r^2)^s\,r\,dr\leqslant C_1,$$

while for $s<\frac{3}{2}$

$$\left|\iint_{|\zeta|>1}(\psi_h^2+|\zeta|^2)^s\,\tilde{\phi}(\zeta)^2\,d\zeta\right|\leqslant C\int_1^{\infty}r^{-4+2s}\,dr\leqslant C_2.$$

Hence (A3) follows. Replacing χ_P by χ_P^2 we get in exactly the same way

$$\|\chi_P^2\|_s\leqslant C\psi_h^{1-s} \qquad \text{for} \quad -1<s<\tfrac{5}{2}.$$

References

1. G. I. Ėskin, "Boundary Value Problems for Elliptic Pseudodifferential Equations," Amer. Math. Soc. Transl. of Math. Monographs (from Russian original), Vol. 52, Amer. Math. Soc. Providence, R.I., 1981.
2. W. A. Heiskanen and H. Moritz, "Physical Geodesy," Freeman, San Francisco, 1967.
3. L. Hörmander, The boundary problems of physical geodesy, *Arch. Rational Mech. Anal.* **62** (1976), 1–52.
4. L. Hörmander, Symbolic calculus and differential equations, *in* "Proceedings 18th Scandinavian Congr. Math.," Birkhäuser, Boston, 1981.

5. W. MAGNUS, F. OBERHETTINGER, AND R. P. SONI, "Formulas and Theorems for the Special Functions of Mathematical Physics," Springer-Verlag, Berlin/Heidelberg/New York, 1966.
6. S. G. MIKHLIN, "Mathematical Physics, an Advanced Course," North-Holland, Amsterdam, 1970.
7. S. L. SVENSSON, "Solution of the Altimetry–Gravimetry Problem," *Bull. Géod.*, to appear.
8. G. SZEGÖ, Uber einige asymptotische Entwicklungen der Legendreschen Funktionen, *Proc. London Math. Soc.* (2) (1934), 427–450.